The MOD notation: $\langle k\rangle_N \equiv$ Remainder when N *divides* k

Periodic functions in terms of mod notation: $\langle k\rangle_N = \langle k+N\rangle_N$

Mod notation for circular shifting: $x[\langle k-m\rangle_N] = \begin{cases} x[n-m] & m \le n \le N-1 \\ x[N-m+n] & 0 \le n < m \end{cases}$

Circular Convolution: $x[n] \circledast g[n] = \sum_{m=0}^{N-1} x[m]\cdot g[\langle n-m\rangle_N]$

Time reversal: $g[n] \coloneqq x[\langle -n\rangle_N] = \begin{cases} x[0] & n=0 \\ x[N-n] & 1 \le n \le N-1 \end{cases}$

Discrete Fourier Transform (DFT):

$\{x[n], 0 \le n \le (N-1)\} \to DFT_N\{\vec{x}\} \to \{X[k], 0 \le k \le (N-1)\}$

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{-kn};\ W_N = \exp\left(j\frac{2\pi}{N}\right);\ 0 \le k \le N-1$$

Inverse DFT: $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{kn};\ 0 \le n \le N-1$

Matrix Representation of DFT

$$\vec{X} = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & W_N^{-kn} \end{bmatrix} \vec{x}; \qquad len(\vec{x}) = len(\vec{X}) = N$$

Periodicity of W_N → $W_N^k = W_N^{\langle k\rangle_N}$ & $W_N^{mk} = W_N^{m\langle k\rangle_N}$

Delta Identity of DFT: In Fourier transform, $\delta(t) \to 1$

$$DFT_N\{\vec{\alpha}\} \to \sum_{n=0}^{N-1} \alpha W_N^{-nk} = \alpha N \delta[\langle k\rangle_N]$$

$DFT_N\{\alpha\delta[\langle n\rangle_N]\} \to N\vec{\alpha} \quad where\ \vec{\alpha} = [\alpha\ \alpha \ldots \alpha]$

DFT Symmetry Properties: (S) := *Symmetric*, (A) := *Anti-symmetric*

Function Prop.	DFT Prop.	Real DFT Prop.	Imag. DFT Prop.
$x \in \mathbb{R}$	$X \in \mathbb{C}$	$X_{\mathbb{R}} \equiv (S)$	$X_{\mathbb{I}} \equiv (A)$
$x \in \mathbb{R}$ & $x_{\mathbb{R}} \equiv (S)$	$X \in \mathbb{R}$	$X_{\mathbb{R}} \equiv (S)$	–
$x \in \mathbb{R}$ & $x_{\mathbb{R}} \equiv (A)$	$X \in \mathbb{I}$	–	$X_{\mathbb{I}} \equiv (A)$
$x \in \mathbb{C}$	$X \in \mathbb{C}$	$X_{\mathbb{R}} \equiv (A)$	$X_{\mathbb{I}} \equiv (S)$
$x \in \mathbb{C}$ & $x_{\mathbb{I}} \equiv (S)$	$X \in \mathbb{I}$	–	$X_{\mathbb{I}} \equiv (S)$
$x \in \mathbb{C}$ & $x_{\mathbb{I}} \equiv (A)$	$X \in \mathbb{R}$	$X_{\mathbb{R}} \equiv (A)$	–

DFT Properties	Time Domain	Frequency Domain
Periodicity	$x[n] = x[\langle n\rangle_N]$	$X[k] = X[\langle k\rangle_N]$
Modulation	$W_N^{mn} x[n]$	$X[\langle k-m\rangle_N]$
Circular Shift	$x[\langle n-m\rangle_N]$	$W_N^{-mk} X[k]$
Circular Convolution	$x[n] \circledast g[n]$	$X[k]\cdot G[k]$
Time Reversal	$x[\langle -n\rangle_N]$	$X[\langle -k\rangle_N]$
Complex Conjugation	$x^*[n]$	$X^*[\langle -k\rangle_N]$
Perceval's Theorem	$\sum_{n=0}^{N-1} x[n]\cdot g^*[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\cdot G^*[k]$	

Real Valued DFT Property: $x[n] \epsilon \mathbb{R} \to X[k] = X^*[\langle -k\rangle_N]$

Real and Circular symmetric: $x[n]\epsilon\mathbb{R}$ & $x[n] = x[\langle -n\rangle_N] \to X[k]\epsilon\mathbb{R}$ & $X[k] = X[\langle -k\rangle_N]$

DFT distribute the function into N points ranging from [0:2π)

Therefore, $X^f(\omega_k) = X\left(\frac{2\pi}{N}k\right)$; $\omega_k \equiv$ Normalized Frequency $= \frac{2\pi}{N}k;\ 0 \le k \le (N-1)$

Physical Frequency of k: $f_k = k\Delta f = k\left(\frac{F_s}{N}\right) = \frac{k}{NT_s}$

DFT Leakage: $x(n) = \cos(\omega_0 n)$ and $S(\omega) = \frac{\sin\left(\frac{N}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$ (digi. sinc function: $\mathrm{DTFT}\{\mathrm{Rect}_N[n]\}$)

$$\mathrm{DTFT}\{x[n]\} \to X^f(\omega) = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$$

$$\mathrm{DTFT}\{x[n]\cdot \mathrm{Rect}_N[n]\} \to \mathrm{DFT}_N\{\vec{x}\} \to \frac{1}{2\pi}\left(X^f(\omega) * S(\omega)\right) = \frac{1}{2}[S(\omega-\omega_0) + S(\omega+\omega_0)]$$

Fast Fourier Transform is based on two properties of DFT:

$W_N^{2nk} = W_{\frac{N}{2}}^{nk}$ & $W_N^{k+\frac{N}{2}} = -W_N^k$. By dividing $x[n]$ into odd (X_1) and even (X_0) samples, we can:

$$X[k] = [k] + W_N^{-k} X_1[k] \quad \& \quad X\left[k+\frac{N}{2}\right] = X_0[k] - W_N^{-k} X_1[k] \quad for\ 0 \le k \le \frac{N}{2}-1$$

FFT Algorithm and Computational Complexity

```
{f is an array of size N=2^p}
FUNCTION DIT(N,f);
LOCAL N',n',fe,fo,Fe,Fo,k,x,F;
IF N==1
   THEN             RETURN(f); {trivial if N==1}
   ELSE             BEGIN      {perform 2 sub-transforms}
        N':=N/2;   {size of sub-transforms}
        FOR n':=0 TO (N'-1) DO
          BEGIN
          fe[n']:= f[2*n'  ]; {even n}
          fo[n']:= f[2*n'+1]; {odd  n}
          END;
        Fe:=DIT(N',fe); {even n}
        Fo:=DIT(N',fo); {odd  n}
        FOR k:=0 TO (N'-1) DO {perform N' DIT 'butterflies'}
          BEGIN
          x=Fo[k]*T(N,k);    {twiddle the odd n results}
          F[k   ]:= Fe[k]+x; {top    subset}
          F[k+N']:= Fe[k]-x; {bottom subset}
          END;
        RETURN(F);
        END;
```

Computational Complexity

DFT: $O(N^2)$ multiplies & N(N − 1) add

FFT: $O(N\log_2 N)$ multiplies

Digital Filters:

$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$ → $y[n] = \sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]$

FIR Digital Filters (When poles are all zero):

$H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}$ → $y[n] = \sum_{k=0}^{M} b_k x[n-k]$

Z Transform: $X(Z) = Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

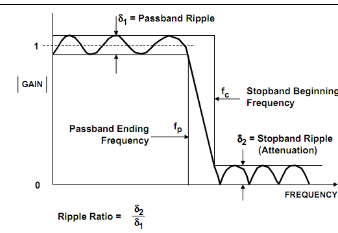
Inverse Z Transform: $x[n] = Z^{-1}\{X(z)\} = \frac{1}{2\pi j}\oint_{\mathbb{C}} X(z) z^{n-1} dz$

Z-Transform Pairs

	Signal, $x(n)$	z-Transform, $X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
3	$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
4	$na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
5	$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
6	$-na^n u(-n-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
7	$\cos(\omega_0 n)u(n)$	$\frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
8	$\sin(\omega_0 n)u(n)$	$\frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
9	$a^n\cos(\omega_0 n)u(n)$	$\frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$
10	$a^n\sin(\omega_0 n)u(n)$	$\frac{1-az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$

Z-Transform Properties

Property	Time Domain	z-Domain	ROC
Notation:	$x(n)$	$X(z)$	ROC: $r_2 < \|z\| < r_1$
	$x_1(n)$	$X_1(z)$	ROC_1
	$x_2(n)$	$X_1(z)$	ROC_2
Linearity:	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least $\mathrm{ROC}_1 \cap \mathrm{ROC}_2$
Time shifting:	$x(n-k)$	$z^{-k}X(z)$	ROC, except $z=0$ (if $k>0$) and $z=\infty$ (if $k<0$)
z-Scaling:	$a^n x(n)$	$X(a^{-1}z)$	$\|a\|r_2 < \|z\| < \|a\|r_1$
Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \|z\| < \frac{1}{r_2}$
Conjugation:	$x^*(n)$	$X^*(z^*)$	ROC
Real part:	$\mathrm{Re}\{x(n)\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Includes ROC
Imaginary part:	$\mathrm{Im}\{x(n)\}$	$\frac{1}{2}j[X(z) - X^*(z^*)]$	Includes ROC
z-Differentiation:	$n\,x(n)$	$-z\frac{dX(z)}{dz}$	$r_2 < \|z\| < r_1$
Convolution:	$x_1(n) * x_2(n)$	$X_1(z)X_2(z)$	At least $\mathrm{ROC}_1 \cap \mathrm{ROC}_2$
Correlation:	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$R_{x_1x_2}(z) = X_1(z)X_2(z^{-1})$	At least $\mathrm{ROC}_1 \cap$ ROC of $X_2(z^{-1})$
Initial value theorem:	If $x(n)$ causal	$x(0) = \lim_{z\to\infty} X(z)$	



Attenuation:

$$A_p = -20\log_{10}(1-\delta_p)$$

$$A_s = -20\log_{10}(\delta_s)$$

$$\delta_p = 1 - 10^{-\frac{A_p}{20}}$$

$$\delta_s = 10^{-\frac{A_s}{20}}$$

Linear Phase FIR Filters: $H^f(\omega) = \mathrm{R}(\omega) + j\mathrm{I}(\omega) = A(\omega)e^{j\theta(\omega)};\ \omega \in \mathbb{R}$

$$\theta(\omega) = -M\omega + B \qquad \frac{\theta(\omega)}{\omega} = K$$

Type	S/A	Len	Properties $A(\omega)$	Auto. Zeros	Usual Types	Not Suitable For
I	S_y	O	$A(\omega)$ is E @ $\omega=0$ $A(\omega)$ is E @ $\omega=\pi$ $A(\omega)$ is Periodic @ 2π	–	All	–
II	S_y	E	$A(\omega)$ is E @ $\omega=0$ $A(\omega)$ is O @ $\omega=\pi$ $A(\omega)$ is Periodic @ 4π	$\omega=\pi$	LPF BPF	BSF
III	A	O	$A(\omega)$ is O @ $\omega=0$ $A(\omega)$ is O @ $\omega=\pi$ $A(\omega)$ is Periodic @ 2π	$\omega=\{0,\pi\}$	Dx Hilbert	LPF,HPF BPF,BSF
IV	A	E	$A(\omega)$ is O @ $\omega=0$ $A(\omega)$ is E @ $\omega=\pi$ $A(\omega)$ is Periodic @ 4π	$\omega=0$	HPF BPF Hilbert	LPF BSF

A: Anti symmetric, S_y: Symmetric, E: Even, O: Odd, B: Band, H: High, L: Low, F: Filter, S: Stop, Dx: Diff.

DFT Based Interpolation for Designing FIR Filter

I & II	$A\left(\frac{2\pi}{L}k\right) = \mathrm{DFT}_L\{[\vec{h}, \vec{0}_{L-N}]\}\cdot W_N^{Mk};\ k = [0:(L-1)]$
III & IV	$A\left(\frac{2\pi}{L}k\right) = -j\,\mathrm{DFT}_L\{[\vec{h}, \vec{0}_{L-N}]\}\cdot W_N^{Mk};\ k = [0:(L-1)]$

FIR Filter Types:

Type	$A(\omega)$	$\theta(\omega)$
I	$h(M)+2\sum_{n=0}^{M-1} h(n)\cos\big((M-n)\omega\big)$	$-M\omega$
II	$2\sum_{n=0}^{M-1} h(n)\cos\big((M-n)\omega\big)$	$-M\omega$
III	$2\sum_{n=0}^{M-1} h(n)\sin\big((M-n)\omega\big)$	$-M\omega+\frac{\pi}{2}$
IV	$2\sum_{n=0}^{\frac{N}{2}-1} h(n)\sin\big((M-n)\omega\big)$	$-M\omega+\frac{\pi}{2}$
$M=\frac{N-1}{2}$		

Ideal Impulse Response Truncation:

$$d[n]=\text{IDTFT}\{D^f(\omega)\}=\frac{1}{2\pi}\int_{-\pi}^{\pi} D^f(\omega)e^{j\omega n}d\omega$$

$$A^f(\omega)=\frac{1}{2\pi}\left(D^f(\omega)\circledast\mathcal{W}^f(\omega)\right);\ \mathcal{W}(\omega)\equiv \text{Window Function}$$

$$\mathcal{W}^f(\omega)=\text{DTFT}\{w[n]\}$$

$$H^f(\omega)=\frac{1}{2\pi}\left(D^f(\omega)\circledast\mathcal{W}^f(\omega)\right)e^{-jM\omega}$$

$$\therefore h[n]=d[n-M]w[n]$$

∗For a fixed length window, there is a tradeoff between main lobe width and relative side lobe height. They may not be made arbitrarily good at the same time.

∗To reduce spectral leakage, a window with low side lobes is required

∗To reduce frequency smearing, a window with narrow main-lobe is required

Non-adjustable windows:

For: $w[n]=a-b\cos\left(\frac{2\pi(n+1)}{N+1}\right)+c\cos\left(\frac{4\pi(n+1)}{N+1}\right)$

Window	a	b	c
Rectangular	1	0	0
Hann	0.5	0.5	0
Hamming	0.54	0.46	0
Blackman-Harris	0.52	0.5	0.8

Zero Location of a REAL VALUED FIR Filter:

- Zeros location of a real valued impulse response exits in sets of 4: $\left\{z_0, z_0^*, \frac{1}{z_0}, \frac{1}{z_0^*}\right\}$
- Zeros on the unit circle ($z_0=e^{j\omega_0}$) exists in sets of two (not $z_0=\pm1$): $\{z_0=e^{\pm j\omega_0}\}$
- Zeros on the real axis ($z_0 = a$) exists in sets of two: $\left\{z_0, \frac{1}{z_0}\right\}$
- Zeros at ±1 does not imply the existence of zeros at other location.

STFT:

Parameters: R: Block length, $w[n]$: Specific window function, L: Number of samples between adjacent blocks, N: FFT length after zero padding.

$$X[\omega,m]=\text{STFT}\{x[n]\}=\text{DTFT}\{x[n-m]w[n]\}=\sum_{n=-\infty}^{\infty}x[n-m]w[n]\exp(-j\omega n)$$

$$X[\omega,m]=\sum_{n=0}^{R-1}x[n-m]w[n]\exp(-j\omega n)$$

$$\Longrightarrow X^d[k,m]=DFT_N\left\{\left[\,[x[n-m]w[n]]_{n=0}^{R-1},\vec{0}_{N-R}\,\right]\right\}$$

Least Square Filter Design:

General Error Formula (L_P Error): $E_P=\int_0^\pi W(\omega)|A(\omega)-D(\omega)|^P d\omega$

$\varepsilon_2=\int_0^\pi W(\omega)\big(A(\omega)-D(\omega)\big)^2 d\omega$; W: non-negative weighting function, A: Amplitude response, D: Desired amplitude response.

In order to minimize square error:

$$\sum_{n=0}^{M}Q[k,n]a[n]=b[k];\qquad 0\le k\le M$$

$$Q[k,n]=\frac{1}{\pi}\int_0^\pi W(\omega)\cos(n\omega)\cos(k\omega)\,d\omega\qquad b[k]=\frac{1}{\pi}\int_0^\pi W(\omega)D(\omega)\cos(k\omega)\,d\omega$$

$$\begin{bmatrix}Q[0,0] & \cdots & Q[0,M]\\ Q[1,0] & \cdots & Q[1,M]\\ \vdots & \ddots & \vdots\\ Q[M,0] & \cdots & Q[M,M]\end{bmatrix}\begin{bmatrix}a[0]\\ a[1]\\ \vdots\\ a[M]\end{bmatrix}=\begin{bmatrix}b[0]\\ b[1]\\ \vdots\\ b[M]\end{bmatrix}\Longrightarrow \vec{a}=\mathbf{Q}^{-1}\vec{b}$$

$$Q[k,n]=\frac{1}{2}Q_1[k,n]+\frac{1}{2}Q_2[k,n]\qquad Q_1[k,n]=\frac{1}{\pi}\int_0^\pi W(\omega)\cos\big((k-n)\omega\big)\,d\omega=q[k-n]$$

$$Q_2[k,n]=\frac{1}{\pi}\int_0^\pi W(\omega)\cos\big((k+n)\omega\big)\,d\omega=q[k+n]\qquad q[n]=\frac{1}{\pi}\int_0^\pi W(\omega)\cos(n\omega)\,d\omega$$

$$q[n]=\text{DTFT}^{-1}\{W(\omega)\}\qquad b[n]=\text{DTFT}^{-1}\{W(\omega)D(\omega)\}$$

Special Case $W(\omega)=1$:

$$a[0]=\frac{1}{\pi}\int_0^\pi D(\omega)d\omega\qquad a[n]=\frac{2}{\pi}\int_0^\pi D(\omega)\cos(n\omega)\,d\omega;\quad 1\le n\le M$$

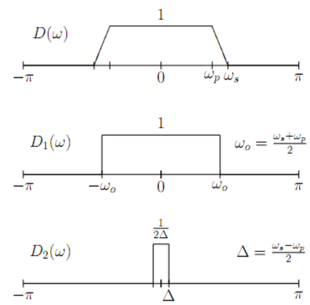
$$h[n]=\frac{1}{\pi}\int_0^\pi D(\omega)\cos\big((n-M)\omega\big)\,d\omega;\qquad 0\le n\le N$$

$\therefore h[n]=d[n-M];\ \ 0\le n\le N$, Where $d[n]=\text{DTFT}^{-1}\{D(\omega)\}$

Ideal Low Pass Filter (LPF):

$$h[n]=\frac{\omega_0}{\pi}\text{sinc}\left[\frac{\omega_0}{\pi}(n-M)\right];\quad 0\le n\le N;\quad \text{sinc}(\theta)=\frac{\sin(\pi\theta)}{\pi\theta}$$

Low Pass Filter with Transition Band:



$$D(\omega)=D_1(\omega)*D_2(\omega)$$

$$d[n]=2\pi\,d_1[n]\,d_2[n]$$

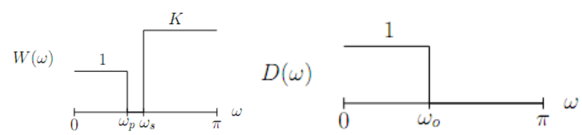
$$d[n]=\frac{\omega_0}{\pi}\text{sinc}\left[\frac{\omega_0}{\pi}n\right]sinc\left[\frac{\Delta}{\pi}n\right]$$

P[th] Order LPF:

$$d[n]=\frac{\omega_0}{\pi}\text{sinc}\left[\frac{\omega_0}{\pi}n\right]\left[sinc\left[\frac{\Delta}{\pi}n\right]\right]^p$$

$$\Delta=\frac{(\omega_p-\omega_s)}{p}$$

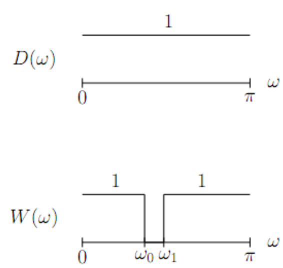
LPF Weighted Square Error:



$$q[k]=\begin{cases}\frac{\omega_p}{\pi}+K\left(1-\frac{\omega_s}{\pi}\right) & k=0\\ \frac{\omega_p}{\pi}sinc\left(\frac{\omega_p}{\pi}k\right)-K\frac{\omega_s}{\pi}sinc\left(\frac{\omega_s}{\pi}k\right) & k\ne0\end{cases}$$

$$b[k]=\frac{\omega_p}{\pi}sinc\left(\frac{\omega_p}{\pi}k\right)$$

Notch Filter:



$$q[k]=\begin{cases}\frac{\omega_0}{\pi}+1-\frac{\omega_1}{\pi} & k=0\\ \frac{\omega_0}{\pi}\text{sinc}\left(\frac{\omega_0}{\pi}k\right)-\frac{\omega_1}{\pi}\text{sinc}\left(\frac{\omega_1}{\pi}k\right) & k\ne0\end{cases}$$

$$b[k]=q[k]$$

Discrete Square Error:

$$W_k=W\left(\frac{2\pi}{L}k\right);\quad D_k=D\left(\frac{2\pi}{L}k\right);\qquad 0\le k\le L-1$$

$$q[n]\approx\frac{1}{L}\sum_{k=0}^{L-1}W_k\exp\left(jn\frac{2\pi}{L}k\right)\ \therefore\ q[n]\approx\text{DFT}_L^{-1}\{W_k\}\quad and\quad b[n]\approx\text{DFT}_L^{-1}\{D_kW_k\}$$

Kaiser Minimum Filter Length Formula:

$$N\approx\frac{-20\log_{10}\left(\sqrt{\delta_s\delta_p}\right)-13}{14.6\Delta F}+1$$